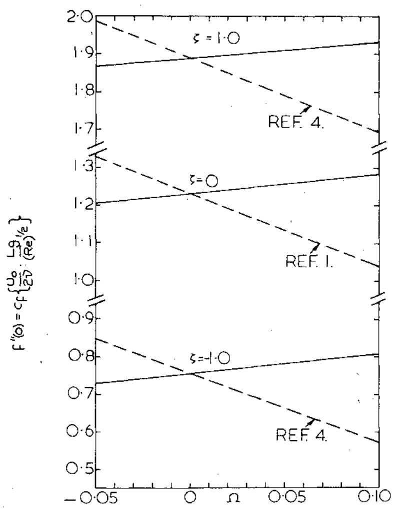

Fig. 2 Variation of $f''(0)$ with Ω and ζ.

in directions parallel to the boundary there is no variation of either thickness or velocity, this solution is not merely "similar" but "identical."

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Numerical Solution for Viscous Transonic Flow

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Introduction

THE near-inviscid supercritical flow over a thin airfoil is calculated using a second-order algorithm for the viscous transonic equation. Type-differencing, shock-point, and parabolic operators are unnecessary in the present approach; good agreement is found with the results of Martin[1] using a Murman-Cole scheme. Some special advantages of the present formulation are discussed.

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Analysis

The computation of steady, inviscid supercritical transonic flow over airfoils is complicated by the appearance of mixed supersonic/subsonic regions separated by unknown sonic lines and shocks. General existence and uniqueness theorems are still unavailable, but a number of successful schemes have been developed recently to handle mixed-type partial differential equations. A number of the methods may be considered "embedding methods" and generally involve computations in a function space other than the physical, the generalized solution of which reduces to the physical solution in some limit. One example is Garabedian's[2] "method of imaginary characteristics," where analytic continuation transforms an unstable elliptic Cauchy problem into a stable hyperbolic one in a complex space, thereby stabilizing the marching procedure. Magnus and Yoshihara,[3] for example, use the method of characteristics to calculate the steady asymptote of the unsteady hyperbolic Euler equations. Still another method is Rubbert's[4] method of parametric differentiation, in which the nonlinear equations are embedded in a parameter space where the governing equations are linear.

The most popular approach is due to Murman and Cole[5] employing type-differencing. Subsonic points are represented by central differences and supersonic points by backward differences, properly accounting for domains of influence and dependence. The manner in which grid points are type-tested is crucial, since divergence in the relaxation scheme is possible. Also, the inviscid finite-difference equations must be in proper conservation form, so that captured shocks with the correct jumps appear. The procedure relies on special parabolic and shock-point operators applied at sonic lines and shocks, and high-order truncation terms must be diffusive. It is clear that sophisticated program logic is required.

A simpler approach is to deal directly with the high-order viscous problem. Consider, for example, Burger's equation, $uu_x = \epsilon u_{xx}$, written in stationary coordinates. Since $(\frac{1}{2}u^2)_{x=a}^{x=b} = (\epsilon u_x)_a^b$, shocklike solutions with vanishing gradients at $x=a,b$ imply straightforwardly the jump conditions $u^2(a) = u^2(b)$. On the other hand, suppose that $\epsilon = 0$ identically, as in the inviscid problem. Then, $uu_x = 0$ can be multiplied by any power of u, with the result that $(u^n)_x = 0$. This admits an infinity of jump conditions, so that one or more entropy conditions must be invoked to insure uniqueness. The jump conditions, of course, are not ambiguous; they are determined by the full, high-order problem.

Similarly, the complicated logic schemes involved in inviscid Murman-Cole-type algorithms can be avoided by embedding the low-order equation in a high-order viscous system. For small-perturbation flows, the vorticity generation within the flow can be ignored, so that the inviscid velocity potential can be used. However, it is known that inviscid theory is not completely sufficient. For example, it inadequately describes the flow near the throat of a converging-diverging nozzle during the transition from the Taylor type of flow to subsonic-supersonic Meyer flow. It also is obvious that real dissipative effects must be important in narrow shock layers. To resolve the physical details near these two kinds of turning points, the usual inviscid derivation must be reconsidered to determine the circumstances under which high-order streamwise derivatives, multiplied by viscosity, are important. This special limiting process was investigated by Sichel,[6] and the result is a "viscous transonic equation" that contains a third derivative term in the disturbance velocity potential with a small coefficient that accounts for the effects of compressive viscosity in shock regions. The modified equation still describes inviscid flow only, but it implicitly contains the correct jump conditions. If accurate solution is possible, all of the salient physical features of flows developed within the framework of the theory would be described. This method is conceptually more attractive because the governing equation is parabolic throughout, rendering type-differencing unnecessary. Furthermore, special shock-point and parabolic operators used in Murman-Cole schemes near shocks and sonic lines, and conservative differencing no longer are needed. The motivation for the present approach was supplied by Reddy,[7] who provided the author with initial results and ideas. We since have studied the relative merits in Reddy's scheme, modifying the original approach in a number of ways, and explained the basic underlying motivation.

Numerical Approach

The nondimensional viscous transonic equation for the perturbation velocity potential is, as given by Sichel,[6]

$$K_v \varphi_{xxx} + (K_\infty - \varphi_x)\varphi_{xx} + \varphi_{yy} = 0 \qquad (1)$$

where $K_v / K_\infty \ll 1$. It is supplemented by the boundary conditions $\varphi_y(x,y) = f'(x)$ on $y=0$, $f'(x)$ being the airfoil slope, with $\varphi_y, \varphi_x \to 0$ as $x^2 + y^2 \to \infty$. The inviscid and viscous similarity parameters are defined by $K_\infty = (1 - M_\infty^2)/[\tau(\gamma+1)M_\infty^2]^{2/3}$ and $K_v = [1 + (\gamma-1)/Pr]/[\tau(\gamma+1)M_\infty^2]^{1/3}$ Re, respectively. Here, M_∞ is the freestream Mach number, τ is half the thickness ratio, γ is the ratio of specific heats, and Pr and Re are Prandtl and Reynolds numbers based on the longitudinal viscosity. The pressure coefficient is given by

$$C_p = -[2\tau^{2/3}/(\gamma+1)^{1/3} M_\infty^{2/3}]\varphi_x \qquad (2)$$

Since $K_v > 0$, Eq. (1) is parabolic, and the characteristics are $y =$ const. This suggests horizontal line relaxation, so that the computation box is swept upwards continually and repeated until convergence. Because of the parabolic nature, it is possible to ignore inviscid subsonic/supersonic type-differencings; the difference formulas are the same everywhere. Since horizontal line relaxation is used, consider lines of constant j, as shown in Fig. 1. Along such lines, a one-dimensional box scheme is employed, as outlined in Keller.[8] Thus, we introduce the three-component vector $(\varphi, \varphi_x, \varphi_{xx}) = (\varphi, U, B)$ at each grid point. The equation $U = \varphi_x$, e.g., is differenced according to $(\varphi_{i,j} - \varphi_{i-1,j})/h_i \cong (U_{i,j} + U_{i-1,j})/2$. $B = U_x$ similarly differenced, as are all x dirivatives. Finally, B, U, and φ satisfy $K_v B_x + (K_\infty - U)U_x + \varphi_{yy} = 0$. Here, φ_{yy} is approximated by central differences about the line j, so that inputs from $j \pm 1$ "feed in" neighboring effects. The resulting formula is used for $j = 3,4,\ldots,j_{\max}-1$, since $j = j_{\max}$ is a boundary. Along $j=2$, which lies adjacent to the airfoil, boundary conditions are incorporated using a second-order accurate formula relating φ at j, $j+1$, and $j+2$ to $\varphi_y(i,j=1) = f'(i)$. The equations so derived are second-order accurate[8] in h_i and second-order accurate in g_j as well. Thus, a coarse mesh can be used to obtain good results, relative to the first-order accurate Murman-Cole scheme. Second-order accuracy in solving Eq. (1) is essential, since $K_v \ll 1$.

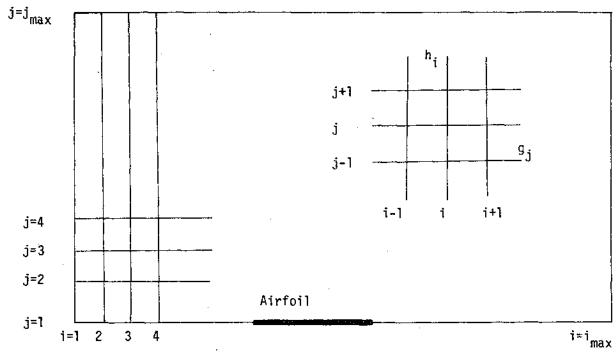

Fig. 1 Computation box and grid system.

Since on each horizontal line there exist $3i_{\max}$ unknowns, the same number of equations is required. This is accomplished by varying the index i from 2 to $i_{\max}$ for each of the previous difference equations. Three additional equations are obtained from boundary conditions imposed at $i = 1$, $i_{\max}$. Homogeneous conditions on φ, φ_x, or φ_{xx} can, of course, be chosen or, perhaps, any linear or nonlinear combination. One form, however, is particularly suggestive. A limited uniqueness theorem for the two-dimensional case gives some indication of properly posed boundary conditions. For a rectangular domain $R(x_1 < x < x_2,\ y_1 < y < y_2)$, it can be shown[6] that, if φ_x is specified on $x = x_1$, and if φ is specified on the entire boundary of R, then the solution to Eq. (1) is unique provided that $\varphi_x < 0$ in R. Note that only one condition is specified on the boundaries parallel to the undisturbed flow, just as in the inviscid case. Also Eq. (1), although parabolic, requires specification of φ over a closed boundary as for second-order elliptic equations. Under these guidelines, we specify φ, φ_x on $i=1$; φ on $i = i_{\max}$; φ on $j = j_{\max}$; and tangency conditions on $j=1$. Homogeneous boundary conditions on box edges are assumed.

Since the full system of equations and boundary conditions is nonlinear along each line $j =$ const, Newton-Raphson iteration (with quadratic convergence) is used. The dependent variable $Z(k,j)$, $k = 1,2,\ldots,3i_{\max}$, $j = 1,2,\ldots,j_{\max}$ is introduced with the definition $U(n,j) = Z(3n-2,j)$, $B(n,j) = Z(3n-1,j)$, and $\varphi(n,j) = Z(3n,j)$ for $n = 1,2,\ldots,i_{\max}$. The difference equations for $B = U_x$ obtained by varying $i = 2,\ldots,i_{\max}$ are numbered as $3,6,9,\ldots,3i_{\max}-3$; those obtained for $U = \varphi_x$ are numbered $4,7,\ldots,3i_{\max}-2$. The remaining equations are numbered $5,8,\ldots,3i_{\max}-1$, respectively; finally, left boundary conditions are numbered 1 and 2, with the right-hand one being $3i_{\max}$. Solution by Newton-Raphson iteration is simplified because the derivative coefficient matrix so ordered takes on a banded structure with bandwidth 7. In each step of the line relaxation and vertical sweeping, the latest updated values of Z are used to evaluate the matrix. The iteration is initialized by a zero perturbation field.

Sample Calculation

Calculations were carried out for a symmetric 10% thick parabolic arc airfoil at Mach 0.820. This flow is known to be supercritical, and comparison with the inviscid conservative differencing/refined variable mesh results of Martin[1] is possible. Twenty uniform grids were assumed over the airfoil and ten for each direction off the airfoil. These off-airfoil horizontal spacings were stretched by a factor of 1.2 each successive grid. Twenty vertical grids were taken. The first two were 2% of chord, and the remainder were stretched by the same 1.2 factor. This resulted in a 4×3 chord computation box. For $\gamma = 1.4$, we have $K_\infty = 1.75$; we further assumed $K_v = 0.0175$. An important parameter that emerges in the analysis is K_v/h^* (h^* being a typical grid size), which

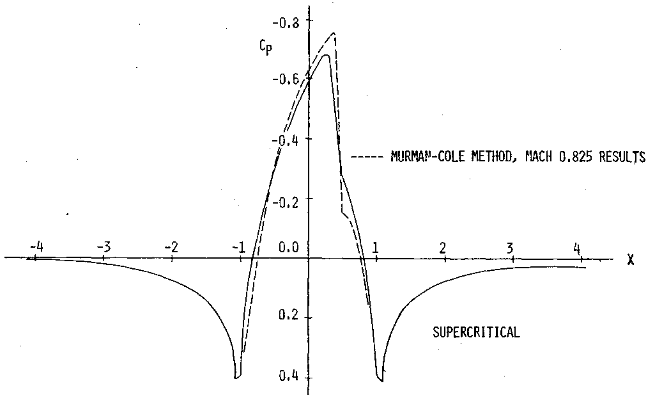

Fig. 2 10% thick parabolic arc airfoil at Mach 0.820 (sweep 650, 40×20 variable grid).

physically measures the viscous length K_v relative to the particle length h^*. It is 0(1) in this example, thus fulfilling the continuum assumption, crucial near shocks. Finally, we indicate that increased values of the viscosity K_v speed convergence (because smoother gradients appear), but these tend also to smear out the shock. Surface pressures ($C_p = -0.231\ \varphi_x$) were calculated by second-order difference formulas.

The preceding problem was run on the CDC 7600 Cybernet System. Convergence was achieved after 500 sweeps of the flowfield, such that surface C_p's differed less than ½%/50 sweeps. Convergence time was estimated at 200 sec. Figure 2 compares the present Mach 0.820 results with 0.825 results for the same airfoil obtained by Martin's[1] improved Murman-Cole algorithm. We note how the extrapolated agreement is excellent, our results being somewhat less peaky, as expected. The magnitude and position of shock jumps and shape of the supersonic zone appear to be well predicted. The shock is smeared over two meshwidths, but this can be reduced by decreasing K_v (at the expense of increasing convergence time).

Discussion

The results obtained from the second-order scheme suggest that solutions obtained in the limit $K_v \to 0$ contain the class of inviscid solutions under Murman-Cole-type jump conditions. The present method, as it stands, however, is not as efficient. The primary reason is the use of Keller's one-dimensional box scheme, which *triples* the number of dependent variables. We thus are pursuing currently this "direct viscosity approach" using a one-dependent-variable second-order differencing scheme. A number of advantages, though, are obvious. Because of second-order accuracy, a coarser mesh can be used to produce Murman-Cole-type results, which are first-order accurate. Secondly, Richardson's extrapolation is possible[8]; thus, good results can be obtained from two relatively coarse nets. Thirdly, the explicit presence of a viscous term permits treatment of non-Rankine-Hugoniot jumps. The major selling point, however, seems to be the simplified program logic. Thus, straightforward extension to the full potential equation with, for example, a simple streamwise diffusion term should be possible. Use of the box scheme in turning-point problems is not new. Our application is analogous to accounting for 0(1) viscous effects near critical layers in direct solutions to the Orr-Sommerfeld equation.

Acknowledgments

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Nonstationarity in Gun Tunnel Flows

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Nomenclature

$\tilde{e}$	= rms value of the fluctuating voltage
e_i	= $\tilde{e}$ at any instant a function of time
$\bar{e}$	= mean value of e_i over a sampling period
f	= frequency
p_0	= stagnation pressure
p_{41}	= gun tunnel driver pressure ratio
T	= sampling period
T_r	= wire recovery temperature
T_W	= hot-wire temperature
t	= time measured from the beginning of the run
v^*	= virtual sensitivity of the hot-wire
ϵ_n	= nonstationary error
ϵ_r	= random error
σ_r	= standard deviation due to randomness
σ_n	= standard deviation due to nonstationarity
Δe_m	= hot-wire mass flow sensitivity
Δe_t	= hot-wire total temperature sensitivity
Δf	= bandwidth

Introduction

TEST facilities such as shock tubes and gun tunnels have been used in recent years for unsteady flow measurement.[1-4] These facilities suffer from the fact that their flows are transient as well as nonstationary; that is, fluctuation signal spectra vary with time. Statistical analysis of the flow unsteadiness in these facilities would involve both the random error due to the statistical averaging process and a nonstationarity error due to the time dependence of the flow. For an analysis of these unsteady flow signals it is necessary to know 1) whether an optimum sampling period exists during the run time for which the total error is within an acceptable limit, and 2) whether a meaningful mean flow condition can be assigned for that sampling period. This Note enlightens these problems in a hypersonic gun tunnel flow.

Tests

Hot-wire anemometer studies were carried out at a point in the exit core flow of a Mach 7 hypersonic jet in a gun tunnel at Loughborough University of Technology. Tests were performed for four wire overheats at $p_0 = 610$ psia and $p_{41} = 10$. The mean voltages were recorded on oscillograms and the fluctuation voltages on a DR channel of an Epsylon MR1200 tape recorder. The DR frequency response was 200 Hz-80 KHz.

Analysis

Recovery temperature during the runs were computed from the hot-wire voltages by the method suggested by Vrebalovich.[5] The extent of nonstationarity in the flow can be seen from the results shown in Fig. 1. The mean tem-

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